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KAN EXTENSIONS OF HOMOTOPY FUNCTORS

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0. Introduction

There are many functors in topology which are most naturally defined on finite or locally finite simplicial complexes. One considers various extensions of these functors to larger classes of spaces, and it is a question as to whether or not the extended functor possesses properties on the larger class of spaces that the original functor did on the smaller.

We will be concerned with such questions as they pertain to two of the standard methods—the topologically oriented method of Čech extension [4, 5, 6, 8, 9, 10] and the categorical Kan extension [8, 12].

The classical examples of the Čech extension are the Čech cohomology functors $\check{H}^n_f$ which extend simplical cohomology from the category of finite simplical complexes to the category of topological spaces. Čech cohomology has most of the usual properties of simplical cohomology with the notable exception that, at least in dimension 1, it is not in general a homotopy invariant for non-compact spaces [9] (Strictly speaking, for non-pseudo compact spaces).

Kan extensions have been studied recently with special regard to the problem of when the Kan extension of a cohomology theory is again a cohomology theory. (See Deleanu and Hilton [5, 6, 7] and Dold [8].) In [6] it is shown that this happens precisely when the Čech and Kan extensions coincide.

Dold, Deleanu and Hilton consider Kan extensions on categories of spaces and homotopy classes of maps. Hence, all functors and their Kan extensions are homotopy functors by definition. Below we show that there is insight to be gained by taking Kan extensions directly over categories of spaces and maps. We observe that in all reasonable cases the Čech extension and the (continuous) Kan extension coincide. Hence, we have available an alternate formulation of the Čech extension. This formulation will be seen to be well suited for studying the relation between the Čech and (homotopy) Kan extension.

The problem that we will be particularly interested in is: "When is the (continuous) Kan (=Čech) extension of a homotopy functor again a homotopy functor?" We will begin with some very general observations about this question and then proceed to specific examples where special information is available.

In section 1 we present a categorical framework for studying questions of this sort. The section begins with a review of the properties of congruences. Next, methods of extending a congruence from one category to a larger category are discussed. In particular, given categories $\mathscr{T}_0 \subseteq \mathscr{T}$ and a congruence R_0 in $\mathscr{T}_0$, we define the codeterminate extension of R_0 to $\mathscr{T}$. This congruence R in $\mathscr{T}$ is shown to be the smallest extension such that the Kan extension of any R_0-functor is an R-functor.

The section ends with a study of the behaviour of codeterminate extensions under reflection.

In section 2 we consider the congruence homotopy. Relations between the homotopy Kan, continuous Kan and Čech extensions are established. For example, it is shown that continuous Kan extensions and Čech extensions agree for the following pairs.

(Top, lf Pol)	locally finite polyhedra
(Comp T₂, f Pol)	Compact T_2, finite polyhedra

and

(C Reg T₂, f Pol)	Completely regular T_2.

Section 3 deals with questions regarding the nature of the extensions of specific functors representable on **f Pol** (i.e. $F=[-, Y]$, homotopy classes of maps). We look at $\check{H}^n_f(-; G)$, $\check{K}O_f$ and $\check{K}_f$, cohomology with G coefficients, real and complex K Theory, respectively.

As already remarked, $\check{H}^1_f$ is not a homotopy functor off pseudo compact spaces [9]. On the other hand, $\check{H}^n_f$ for $n>1$ is a homotopy functor on **fd Norm**, the category of finite dimensional normal spaces [1] and $\check{H}^2_f$ is a homotopy functor on **C Reg T₂** [2]. Here we prove the following:

Theorem. *On* **C Reg T₂**

(a) *For G, an abelian group, $\check{H}^1_f(-; G)$ is a homotopy functor, if and only if G is torsion, that is if G has no element of infinite order.*

(b) *$\check{H}^{2n+1}_f(-, G)$ is not a homotopy functor on countable polyhedra, $n>0$. A counterexample is the simplical path space of S^{2n+1}.*

(c) *$\check{H}^{2n}_f(-; Q)$, Q the rationals, is a homotopy functor.*

(d) *$\check{K}_f$ and $\check{K}O_f$ are homotopy functors.*

In fact, our positive results are somewhat stronger in that in (c) and (d) Puppe sequences are shown to be carried into long exact sequences. Moreover, when the Čech extension is a homotopy functor, then it must agree with the homotopy Kan extension. This gives an alternate formulation of the results of Deleanu and Hilton [6].

Finally, we would like to express our thanks to Alex Heller for his careful reading of this paper, and in particular for his help in developing a suitable context in which to present the material of section 1.

1. Congruences and Kan extensions

In this section we offer a general context for discussing questions of extending congruences on one category to a larger category. We begin with some notation that will be fixed throughout the section.

Let $\mathcal{T}$ be a category. Suppose that for each $X, Y \in \mathrm{ob}\,\mathcal{T}$ we are given a relation $R(X, Y) \subseteq \mathcal{T}(X, Y) \times \mathcal{T}(X, Y)$. We write fRg for $\langle f, g\rangle \in R(X, Y)$. Also, given relations R and R' in $\mathcal{T}$ we say R is finer than R' and write $R \subseteq R'$ if for each pair (X, Y) we have that $R(X, Y) \subseteq R'(X, Y)$.

1.1. Definition [12]. R is called a *congruence* in $\mathcal{T}$ if

(1) For each $X, Y \in \mathrm{ob}\mathcal{T}$, $R(X, Y)$ is an equivalence relation

(2) Given $f: X' \to X$, $g: Y \to Y'$ and $h_1, h_2: X \to Y$ with $h_1 R h_2$ we have that $gh_1 f R gh_2 f$.

In application we will consider the congruence homotopy in various subcategories of **Top**. We will also make use of various constructions through which we form the particular congruences that we will be interested in. The following series of definitions and lemmas form the basis of these constructions.

1.2. Definition. (a) Given a congruence R in $\mathcal{T}$ we form the category $\mathcal{T}/R$ by setting $\mathrm{ob}\,\mathcal{T}/R = \mathrm{ob}\,\mathcal{T}$ and setting $\mathcal{T}/R(X, Y) = \mathcal{T}(X, Y)/R(X, Y)$.

One has the *quotient functor* $Q_R : \mathcal{T} \to \mathcal{T}/R$.

(b) Given a functor $F: \mathcal{T} \to \mathcal{T}'$ define a *congruence* R_F in $\mathcal{T}$ by setting $fR_F g$ if and only if $F(f) = F(g)$.

Finally, combining (a) and (b) we have

(c) Given R' a congruence in $\mathcal{T}'$ and $F: \mathcal{T} \to \mathcal{T}'$ we have $F^{-1}(R') = R_{Q_{R'}\cdot F}$, a *congruence* in $\mathcal{T}$.

Note that in general one has $F^{-1}(R_G) = R_{GF}$.

$\mathcal{T}/R$ is universal for R in the following sense.

1.3. Lemma. [12] *If R is a congruence in $\mathcal{T}$ then $R_{Q_R} = R$.*

Moreover, given a functor $F: \mathcal{T} \to \mathcal{T}'$ with $R \subseteq R_F$ there is a unique functor $\bar{F}: \mathcal{T}/R \to \mathcal{T}'$ such that $\bar{F}Q_R = F$.

The following simple observation is the basis for the main constructions in this section.

1.4. Lemma. *Let $\{R_\alpha\}$ be a class of congruences in $\mathcal{T}$. Define $R(X, Y) = \bigcap R_\alpha(X, Y)$. Then R is a congruence.*

We make use of Lemma 1.4 by observing that given any property of congruences preserved under arbitrary intersection, there is a finest congruence possessing that property. For example, given any relation R_0 there is a finest congruence R containing it. To construct R we let $\{R_\alpha\}$ be the class of congruences such that $R_0 \subseteq R_\alpha$. Note that $\{R_\alpha\}$ is not empty since it contains the congruence that relates all pairs of coterminous morphisms.

We now give that application of Lemma 1.4 with which we shall be most concerned. We begin with a preliminary definition.

1.5. Definition. Given R, a congruence in $\mathcal{T}$, and $\mathbf{A} \subseteq \mathrm{ob}\,\mathcal{T}$ we say that $\mathbf{A}$ *codetermines* R if $\mathbf{A}$ cogenerates $\mathcal{T}/R$ [12] (equivalently fRg if and only if for all $\pi: Y \to P$ with $P \in \mathbf{A}$ we have that $\pi f R \pi g$).

Note, again we can apply Lemma 1.4 and find for any relation R_0 and any $\mathbf{A} \subseteq \mathrm{ob}\,\mathcal{T}$ the finest congruence containing R and codetermined by $\mathbf{A}$. We will denote this congruence by $R_{\mathbf{A}}$.

We are most interested in a special case of 1.5.

1.6. Definition. Given categories $\mathcal{T}_0 \subseteq \mathcal{T}$ and R_0 a congruence in $\mathcal{T}_0$, we will call the finest congruence in $\mathcal{T}$, containing R_0 and codetermined by $\mathrm{ob}\,\mathcal{T}_0$, the *codeterminate extension* of R_0. We will denote it simply by R when no confusion is possible.

In section 2 we will compute the codeterminate extension of homotopy in various topological settings. For our purposes, the importance of these computations lies in the category theoretic relation of the codeterminate extension of R_0 to the (right) Kan extension of certain functors on $\mathcal{T}_0$ to functors on $\mathcal{T}$. We now develop that relation. We begin with a constructive definition of R.

1.7. Theorem. *Let R be the codeterminate extension of R_0. Let $X, Y \in \mathrm{ob}\,\mathcal{T}$ and $f, g: X \to Y$ then fRg if and only if for every $P \in \mathrm{ob}\,\mathcal{T}_0$ and $\pi: Y \to P$ there exists an $n \geq 1$, $Q_1, \ldots, Q_n \in \mathrm{ob}\,\mathcal{T}_0$, and morphisms $\phi_i: X \to Q_i$ and $\pi_{i,j}: Q_i \to P$ $(j = 0, 1$*

and $\pi_{i,j} \in \text{mor}\, \mathcal{T}_0$) *such that*

1) $\pi f = \pi_{1,0}\phi_1, \quad \pi g = \pi_{n,1}\phi_n$

2) $\pi_{i,1}\phi_i = \pi_{i+1,0}\phi_{i+1} \quad 0 \leq i \leq n-1$

3) $\pi_{i,0} R_0 \pi_{i,1} \qquad 0 \leq i \leq n.$

Proof. Let $\bar{R}$ be the relation so determined. Clearly $f\bar{R}g$ implies $fR_\alpha g$ for any congruence in $\mathcal{T}$ such that $R_0 \subseteq R_\alpha$ and R_α is codetermined by ob $\mathcal{T}_0$. Hence $\bar{R} \subseteq R$.

To show $R \subseteq \bar{R}$: one verifies that $R_0 \subseteq \bar{R}$, $\bar{R}$ is a congruence in $\mathcal{T}$ and finally $\bar{R}$ is codetermined by ob $\mathcal{T}_0$. Each of these verifications is straightforward.

For example, to show $f\bar{R}g$ implies $hf\bar{R}hg$ for any $h: Y \to Y'$, let $\pi': Y' \to P$ be arbitrary. Let $\pi = \pi' h: Y \to P$. Since $f\bar{R}g$ we are able to find $(n, \{Q_i\}, \{\phi_i\}, \{\pi_{i,j}\})$ satisfying 1), 2) and 3) for $(f, g\pi)$. Observe that these same data satisfy 1), 2) and 3) for (hf, hg, π').

1.8. Corollary. *Let $\mathcal{T}_0$ be a full subcategory of $\mathcal{T}$. Let R be the codeterminate estension of R_0. Then $R \mid \mathcal{T}_0 = R_0$.*

Proof. We need $\mathcal{T}_0$ to be full in order to be assured that $\{\phi_i\} \subseteq \text{mor}\, \mathcal{T}_0$.

We are now in a position to relate the notion of Kan extension to that of codeterminate extension.

By the Kan extension of a functor $F: \mathcal{T}_0 \to \mathcal{A}$ we will usually have in mind the pointwise right Kan extension as defined in [12, p. 233]. It will be denoted by $F^K: \mathcal{T} \to \mathcal{A}$.

The category $\mathcal{A}$ will always be assumed to be complete and the comma categories $(X \downarrow \mathcal{T}_0)$ will be assumed to have small strongly cofinal subcategories. Thus, in particular, we will assume that F^K always exists.

In our applications $\mathcal{T}$ will be a category of topological spaces, For each $X \in \text{ob}\, \mathcal{T}$ let $\mathcal{T}_0'$ be the full subcategory of $\mathcal{T}_0$ whose objects are spaces of cardinality less than or equal to that of X, $(X \downarrow \mathcal{T}_0')$ is strongly cofinal in $(X \downarrow \mathcal{T}_0)$ and has a small dense subcategory. Then the above conditions on $\mathcal{T}$ will always be present.

1.9. Definition. Given a congruence R in $\mathcal{T}$, a functor $F: \mathcal{T} \to \mathcal{A}$ is called on R-functor if it factors through $\mathcal{T}/R$. ($F = F' Q_R$ where $F': \mathcal{T}/R \to \mathcal{A}$).

1.10. Theorem. *Let $F: \mathcal{T}_0 \to \mathcal{A}$ be an R_0-functor. Then R_{F^K} is codetermined by ob $\mathcal{T}_0$ and $R_{F^K} \supseteq R_0$. Hence $R_{F^K} \supseteq R$.*

(In particular, the Kan extension of an R_0-functor is an R-functor.)

Proof. Suppose f, $g: X \to Y$ are morphisms such that $F^K(\pi f) = F^K(\pi g)$ for all $\pi: Y \to P$, $P \in \mathrm{ob}\, \mathscr{T}_0$. Since $F^K(Y)$ is the vertex of a universal cone to F for $(Y \downarrow \mathscr{T}_0)$ this implies that $F^K(f) = F^K(g)$. Hence R_{F^K} is codetermined by ob $\mathscr{T}_0$ and, since $F^K \mid \mathscr{T}_0 = F$, $R_{R^K} \supseteq R_0$.

We shall also need the following explicit computation.

1.11. Theorem. *Let $\mathscr{T}_0$ be a full subcategory of $\mathscr{T}$. Let $P \in \mathrm{ob}\, \mathscr{T}_0$. Let F_P denote the* **Set**$^{\mathrm{op}}$ *valued functor $\mathscr{T}_0/R_0(-, P)$ on $\mathscr{T}_0$. Then the Kan extension of F_P to $\mathscr{T}$ is $\mathscr{T}/R(-, P)$.*

Proof. (See for example [8, A3.14].) Denote $\mathscr{T}/R(-, P)$ by F'_B. Since $R/\mathscr{T}_0 = R_0$ (1.8) we have that $F'_P \mid \mathscr{T}_0 = F_p$. Hence there is a unique natural transformation $\sigma: F'_P \to F^K_P$ extending the identity on $\mathscr{T}_0$.

To complete the proof we define an inverse $\eta: F^K_P \to F'_P$. We define $\eta_x: F'_P(X) \to F^K_P(X)$ (in **Set**) as follows: If $f: X \to P$ and $Id: P \to P$ is the identity let $\eta_X([f]) = F^K_P(f)][\mathrm{Id}]$. It is immediate that η is natural hence $\sigma\eta = 1$. On the other hand

$$\begin{aligned}
\sigma_X \eta_X [f] &= \sigma_X F^K_P(f)[\mathrm{Id}] \\
&= F'_P(f) \sigma_P [\mathrm{Id}] \\
&= F'_P(f)[\mathrm{Id}] \\
&= [f]
\end{aligned}$$

so $\eta\sigma = 1$.

Combining 1.10 and 1.11 we have the following.

1.12. Theorem. *Let $\mathscr{T}_0$ be a full subcategory of $\mathscr{T}$. Let $\{F_\alpha\}$ be the class of R_0-functors. Let $R^K = \bigcap R_{F^K_\alpha}$. Then $R^K = R$.*

Proof. By 1.10 we have that $R \subseteq R^K$. Conversely if $f \not R g$ then for some $\pi: Y \to P$ $\pi f \not R \pi g$. Hence, by 1.11 $F^K_P(f) \neq F^K_P(g)$ and thus $R^K \subseteq R$.

It will be convenient at times to use the notation R^K for the codeterminate extension.

We shall also be interested in comparing Kan extensions computed over different categories. More specifically, let $\bar{R}$ be a congruence on $\mathscr{T}$. Let $R_0 = \bar{R} \mid \mathscr{T}_0$ and let $F: \mathscr{T}_0 \to \mathscr{A}$ be an R_0-functor, thus $F = \bar{F} Q_{R_0}$. We will wish to know the relation between F^K and $\bar{F}^K Q_{\bar{R}}$ ($\bar{F}^K$ is a Kan extension of $\bar{F}$ to $\mathscr{T}/\bar{R}$). The following theorem gives the information we will need.

1.13. Theorem. *F^K is an $\bar{R}$-functor if and only if $F^K = \bar{F}^K Q_{\bar{R}}$, up to natural equivalence. Thus if R is the codeterminate extension of R_0 and $\bar{R} \subseteq R$ then $F^K = \bar{F}^K Q_{\bar{R}}$.*

Proof. If $F^K = \bar{F}^K Q_{\bar{R}}$ then by definition F^K is an $\bar{R}$-functor. Conversely, let $F^K = \tilde{F} Q_{\bar{R}}$. Then there is a unique natural transformation $\rho : \tilde{F} \to \bar{F}^K$ with $\rho | (\mathcal{T}_0/\bar{R})$ the identity .Also there is a unique natural transformation. $\sigma : \bar{F}^K Q_{\bar{R}} \to F^K$ with $\sigma | \mathcal{T}_0$, the identity. Thus, $\sigma(\rho \circ Q_R) : F^K \to F^K$ is the identity on $\mathcal{T}_0$ hence a natural equivalence.

On the other hand every natural transformation of $\bar{R}$-functors factors uniquely through $Q_{\bar{R}}$. Thus we have $\tilde{\sigma} : \bar{F}^K \to \tilde{F}$ with $\tilde{\sigma} \circ Q_{\bar{R}} = \sigma$. Hence $\rho\tilde{\sigma}$ is a natural equivalence as is

$$(\rho\tilde{\sigma}) \circ Q_{\bar{R}} = (\rho \circ Q_{\bar{R}})(\tilde{\sigma} \circ Q_{\bar{R}}) = (\rho \circ Q_{\bar{R}})\sigma.$$

To complete the proof we note that by 1.10 F^K is an R-functor and since $\bar{R} \subseteq R$ it is then also an $\bar{R}$ functor.

Finally, we will be concerned with the following situation. Let $\mathcal{P}_0 \subseteq \mathcal{P} \subseteq \mathcal{T}$ be three categories. Suppose we are given a reflection, $B : \mathcal{T} \to \mathcal{P}$, of the inclusion $I : \mathcal{P} \to \mathcal{T}$, a congruence R_0 and $\mathcal{P}_0$ and an R_0-functor F. We will wish to know the relation between Kan extension of F to $\mathcal{P}$, and to $\mathcal{T}$, as well as the relation between the respective codeterminate extensions. The specific example we have in mind is $\mathbf{f\,Pol} \subseteq \mathbf{Comp\,T_2} \subseteq \mathbf{C\,Reg\,T_2}$, $\beta : \mathbf{C\,Reg\,T_2} \to \mathbf{Comp\,T_2}$ (Stone–Čech compactification) and R_0 being homotopy on $\mathbf{f\,Pol}$.

We have the following

1.14. Theorem. *Let $\mathcal{P}_0$ be a full subcategory of $\mathcal{T}$. Let F^K be a Kan extension of F to $\mathcal{P}$ and let R be the codeterminate extension of R_0 to $\mathcal{P}$. Then*

(a) *$F^K B$ is a Kan extension of F to $\mathcal{T}$; and*

(b) *the codeterminate extension of R_0 to $\mathcal{T}$ is $B^{-1}(R)$.*

Proof. (a) Let $X \in \operatorname{ob} \mathcal{T}$ and $r_X : X \to B(X)$ its reflection into $\mathcal{P}$. Since r_X is $\mathcal{P}$-universal (hence $\mathcal{P}_0$-universal) it induces a natural equivalence of comma categories $r_x^{\#} : (B(X) \downarrow \mathcal{P}_0) \to (X \downarrow \mathcal{P}_0)$ so that $F^K B(X)$ is the vertex of a universal cone to F over $(X \downarrow \mathcal{P}_0)$. Since $F^K B$ is an extension of F to $\mathcal{T}$ it is accordingly a Kan extension.

(b) Let $F : \mathcal{P}_0 \to \mathcal{A}$. For the purposes of the proof let $\tilde{F}^K$ denote Kan extension to $\mathcal{T}$, let F^K denote Kan extension to $\mathcal{P}$, and let $\tilde{R}$ denote the codeterminate extension of R_0 to $\mathcal{T}$.

By 1.12 we have that $\tilde{R} = \bigcap R_{\tilde{F}_\alpha^K}$ where $\{F_\alpha\}$ is the class of R_0-functors. Therefore by (a)

$$\tilde{R} = \bigcap R_{F_\alpha^K B} = \bigcap B^{-1}(R_{F_\alpha^K}) = B^{-1}\left(\bigcap R_{F_\alpha^K}\right) = B^{-1}(R).$$

2. Homotopy

Let $\mathscr{P}$ be a full subcategory of **Pol**. Let $\mathscr{T} \supseteq \mathscr{P}$ be a full subcategory of **Top**. Below, we apply the results of the previous section to homotopy considered as a congruence in $\mathscr{P}$. We compute the codeterminate extension of homotopy to $\mathscr{T}$ under various hypotheses on $\mathscr{T}$ and $\mathscr{P}$. Our results require certain geometric information which we now obtain, based on the following variant of a definition of Lee and Reymond [11].

2.1. Definition. $(\mathscr{T}, \mathscr{P})$ is called a *Čech extension pair* if:

1) $\mathscr{P} \subseteq$ **Pol**

2) If $P \in \mathrm{ob}\,\mathscr{P}$ then $\hat{P} = \{(z_1, z_2) \in P \times P : z_1$ and z_2 are contained in a single simplex of $P\}$, with the weak topology, is in $\mathscr{P}$.

3) For each $X \in \mathrm{ob}\,\mathscr{T}$, $K \in \mathrm{ob}\,$**Pol** and map $\pi : X \to K$ there is a $P \in \mathrm{ob}\,\mathscr{P}$ and maps $\pi' : X \to P$, $\pi' : P \to K$ such that $\pi \simeq$ (is contiguous) to $\pi''\pi'$. (i.e. For all $x \in X$, $\pi(x)$ and $\pi''\pi'(x)$ lie in a single simplex of K).

Our definition of Čech extension pair differs from that of Lee and Raymond principally in that we are restricting to numerable (=normal) covers, as the following lemma makes clear.

2.2. Lemma. *Let $\mathscr{P}$ satisfy* 1) *and* 2) *of* (2.1), *then $(\mathscr{T}, \mathscr{P})$ is a Čech extension pair, if and only if, for each $X \in \mathrm{ob}\,\mathscr{T}$ we have that every numerable open cover of X is refined by a numerable open cover whose nerve is in $\mathscr{P}$.*

Proof. By standard bridge map arguments of the kind, for example, found in [8, p. 356].

2.3. Examples of Čech extension pairs

(a) **(Top, Pol)**

(b) **(Top, lf Pol)**

(c) **(Comp T_2, f Pol)**

(d) **(SM, clf Pol)**, separable metric, countable locally finite polyhedra.

(e) **(fd Norm, fd Pol)**, finite dimensional normal, finite dimensional polyhedra.

We will be concerned with the codeterminate extension of the homotopy for Čech extension pairs. We denote homotopy by h but will write $f \sim g$ rather than fhg. By a homotopy functor we mean an h-functor.

Homotopy over $\mathscr{P}$. Writing $h_{\mathscr{P}}$ for $h_{\mathrm{ob}\mathscr{P}}$ we have that $h \subseteq h_{\mathscr{P}}$ but in general $h_{\mathscr{P}} \neq h$. For example, take X to be a one point space and Y any connected non-path-connected space, and $f, g : X \to Y$ to be maps such that $f(X)$ and $g(X)$ are in different path components. Then $f \not\simeq g$ but if the objects of $\mathscr{P}$ are polyhedra then $f h_{\mathscr{P}} g$.

Denote the codeterminate extension of h from $\mathscr{P}$ to $\mathscr{T}$ by h^K. $h^K \subseteq h_{\mathscr{P}}$ but again, in general, $h^K \neq h_{\mathscr{P}}$ (see below). However one has

2.4. Theorem. *For $(\mathscr{T}, \mathscr{P})$ a Čech extension pair, $h^K = h_{\mathscr{P}}$.*

Proof. For the proof of (2.4) we need the following Lemma:

2.5. Lemma. *Let $(\mathscr{T}, \mathscr{P})$ be a Čech extension pair. For each $X \in \mathrm{ob}\,\mathscr{T}$, $P \in \mathrm{ob}\,\mathscr{P}$, and homotopy $H : X \times I \to P$ there exists $Q \in \mathrm{ob}\,\mathscr{P}$ and maps $\pi : X \to Q$ and $\theta : Q \times I \to P$ such that $\theta(\pi \times Id) \simeq H$.*

Hence, if H is base point preserving Q may be chosen to be so also.

Proof. This proof is essentially an observation on [8] p. 357 to which the reader is referred for definitions and notation.

Let $\mathscr{W}$ be the cover of P by open stars of vertices. Then $H^{-1}\mathscr{W} = \{H^{-1}W : W \in \mathscr{W}\}$ is a numerable cover of $X \times I$ so there is a stacked cover $\mathscr{U} \times \mathscr{S}$ of $X \times I$ refining $h^{-1}\mathscr{W}$. Let $\{\pi_U^{\mathscr{U}} : X \to I\}$ be a locally finite numeration of $\mathscr{U}$ and for each $U \in \mathscr{U}$ let $\{\pi_V^{\mathscr{S}U} : I \to I\}$ be a locally finite numeration of $\mathscr{S}U$.

Define $\pi^{\mathscr{U}} : X \to \nu\mathscr{U}$ (the nerve of $\mathscr{U}$) by $\pi^{\mathscr{U}}(x) = \{\pi_U^{\mathscr{U}}(x)\}$ = the point whose U-th barycentric coordinate is $\pi_U^{\mathscr{U}}(x)$. Define $\pi^{\mathscr{S}} : \nu\mathscr{U} \times I \to \nu(\mathscr{U} \times \mathscr{S})$ by $\pi^{\mathscr{S}}(\{x_U\}, t) = \{x_U \cdot \pi_V^{\mathscr{S}U}(t)\}$. For each $U \in \mathscr{U}$ and $V \in \mathscr{S}\mathscr{U}$, let w be a vertex of P such that $H(U \times V) \subset \mathrm{star}\, w$. Define $\theta' : \nu(\mathscr{U} \times \mathscr{S}) \to P$ by putting $\theta'(u \times v) = w$ and extending linearly.

Since (T, P) is a Čech extension pair there exists $Q \in \mathrm{ob}\,\mathscr{P}$ and maps $\pi : X \to Q$ and $\pi' : Q \to \nu\mathscr{U}$ such that $\pi^{\mathscr{U}} \simeq \pi'\pi$. Finally define $\theta = \theta'\pi^{\mathscr{S}}(\pi' \times Id)$. It is easily checked that $\theta(\pi \times Id) \simeq H$.

2.6. Proof of 2.4. Again, $h_{\mathscr{P}} \supseteq h^K$ so we need only show that $h^K \supseteq h_{\mathscr{P}}$.

Suppose $f, g : X \to Y$ are homotropic over $\mathscr{P}$ and $\pi : Y \to P$, $P \in \mathrm{ob}\,\mathscr{P}$ is a map. Let $H : X \times I \to P$ be a homotopy from πf to πg. By (2.5) there exists $Q \in \mathrm{ob}\,\mathscr{P}$ and maps $\pi' : X \to Q$ and $\theta : Q \times I \to P$ such that $\theta(\pi' \times Id) \simeq H$.

Let $\hat{P}$ be as in (2.1) (2). Let p_1 and $p_2 : \hat{P} \to P$ be the projections on the first and second factors respectively. Then $p_1 \simeq p_2$. Define $\hat{\pi}_i : X \to \hat{P}$, $i = 1, 2$ by $\hat{\pi}_1(x) = (\pi f(x), \theta(\pi'(x), 0))$ and $\hat{\pi}_2(x) = (\theta(\pi'(x), 1), \pi g(x))$.

Taking $Q_1 = Q_3 = \hat{P}$ and $Q_2 = Q$ in (1.7) gives $f h^K g$.

2.7. Corollary. *If $(\mathscr{T}, \mathscr{P})$ is a Čech extension pair and $F : \mathscr{P} \to \mathscr{A}$ is a homotopy functor then $F^K : \mathscr{T} \to \mathscr{A}$ is a homotopy functor.*

We are also able to combine 2.4 with 1.13 to determine the relationship between the continuous and the homotopy Kan extensions for Čech extension pairs.

2.8. Theorem. *If $(\mathcal{T}, p)$ is a Čech extension pair and $F: \mathcal{P} \to \mathcal{A}$ is a homotopy functor ($F = \tilde{F}Q_h$) then $F^K = \tilde{F}^K Q_h$.*

In other words, for Čech extension pairs our Kan extension of a homotopy functor agrees with the usual one ([5], [8]).

We now consider cases where we do not have a Čech extension pair and in particular, the classical case of extending from **f Pol** to categories larger than **Comp T₂**. Our main result is that with $\mathcal{P} \subset \mathbf{f\,Pol}$ and $\mathcal{T} \subset \mathbf{C\,Reg\,T_2}$, instead of homotopy extending to homotopy over $\mathcal{P}$, it extends to uniform homotopy over $\mathcal{P}$.

Let $\beta : \mathbf{C\,Reg\,T_2} \to \mathbf{Comp\,T_2}$ be the Stone–Čech compactification functor. That is, the functor which sends every space X to its Stone–Čech compactification βX and every map $f: X \to Y$ to the unique extension $\beta(f): \beta X \to \beta Y$.

2.9. Definition. Two maps $f, g : X \to Y$ are called *uniformly homotopic*, if there is a homotopy between f and g which extends to a homotopy between $\beta(f)$ and $\beta(g)$ (for equivalent definitions see [2, 10 p. 282]). We write $f \sim_\beta g$. Of course, $\beta(f) \sim \beta(g)$ does not imply $f \sim g$ but if Y is compact, then $f \sim_\beta g$ if and only if $\beta(f) \sim \beta(g)$. Uniform homotopy defines a congruence on any $\mathcal{T} \subset \mathbf{C\,Reg\,T_2}$.

2.10. Theorem. *Let $(\mathcal{T}, \mathcal{P})$ be such that $\mathcal{T} \subset$ **C Reg T₂**, $\mathcal{P} \subset$ **f Pol** and $(\beta(\mathcal{T}), \mathcal{P})$ is a Čech extension pair. Then*

(a) *For every functor $F : \mathcal{P} \to \mathcal{A}$, we have that $F^K\beta$ is a Kan extension to $\mathcal{T}$*

(b) *The codeterminate extension of homotopy is uniform homotopy over $\mathcal{P}$. That is $h^K = (\beta h)_{\mathcal{P}}$.*

In particular this holds for (**C Reg T₂, f Pol**).

Proof. The theorem is an immediate corollary of 1.14 and 2.4 and the fact that β is a reflection.

We are now in a position to consider the relations between Kan extensions and Čech extensions.

We begin by recalling the definition of Čech extension.

For $X \in \operatorname{ob} \mathcal{T}$, let $\operatorname{Cov}(X, \mathcal{P})$ denote the category whose objects are the numerable covers of X with nerves in $\mathcal{P}$. For any pair of objects $\mathcal{U}$ and $\mathcal{V}$ we have a unique morphism $\mathcal{U} < \mathcal{V}$ if and only if, $\mathcal{U}$ is a refinement of $\mathcal{V}$.

With $\tilde{\mathcal{P}}$ the homotopy category of $\mathcal{P}$, we define a functor $\theta_X : \operatorname{Cov}(X, \mathcal{P}) \to \tilde{\mathcal{P}}$ by letting $\theta_X(\mathcal{U} < \mathcal{V})$ be the homotopy class of a canonical projection $\pi^{\mathcal{V}}_{\mathcal{U}} : \nu\mathcal{U} \to \nu\mathcal{V}$.

If $F : \mathcal{P} \to \mathcal{A}$ is a homotopy functor ($F = \tilde{F}Q_h$) we define the Čech extension $\check{F} : \mathcal{T} \to \mathcal{A}$ of F by $\check{F}(X) = \varprojlim \tilde{F}\theta_X$.

2.11. Theorem (Dold). *If (T, P) is a Čech extension pair and $F : \mathcal{P} \to \mathcal{A}$ is a homotopy functor then $\check{F} = F^K$.*

Proof. By (2.8), $F^K = \tilde{F}^K Q_h$ and by [8 p. 366] $\tilde{F}^K Q_h = \check{F}$. (In fact, Dold only proved this for (**Top**, **Pol**); but, as he remarks, the proof goes through for all Čech extension pairs).

Finally we have

2.12. Theorem. *If $\mathcal{P} \subset$ **f Pol**, $\mathcal{T} \subset$ **C Reg T₂** and $(\beta(\mathcal{T}), \mathcal{P})$ is a Čech extension pair, then for every homotopy functor $F : \mathcal{P} \to \mathcal{A}$, $\check{F} = F^K$.*

Proof.

$$F^K = F^K\beta \quad \text{by (2.10) (a)}$$

$$F^K\beta = \check{F}\beta \quad \text{by (2.11)}$$

and

$$\check{F}\beta = \check{F} \quad \text{by [10 p. 282].}$$

Given that (2.11) and (2.12) include all the situations where the Čech extensions are usually considered; it would appear that the proper way to view Čech extensions is as Kan extensions.

Note that in general for non-Čech extension pairs $\check{F}$ is not a homotopy functor and so $\check{F} \neq \tilde{F}^K Q_h$.

The pair (**C Reg T₂**, **f Pol**) also provides an example of a congruence R for which neither $R \subseteq (R \mid \mathcal{P})^K$ nor $(R \mid \mathcal{P})^K \subseteq R$. On the one hand maps may be homotopic without being uniformly homotopic over **f Pol**. On the other hand, one may choose maps f and g such that $f \not\sim g$ yet $\beta(f) \sim \beta(g)$. Hence, f and g are uniformly homotopic over **f Pol** without being homotopic.

3. Čech extensions of representable functors

In this section we restrict attention to categories of base pointed spaces and maps. Letting **p Set** be the category of pointed sets and functions, $[X, Y] \in \text{ob}$ **p Set** (**p Set**$^{\text{op}}$) will denote the set of base point preserving homotopy classes of maps.

The results of the previous section together with 1.11 give us the following.

3.1. Theorem. *Let $Y \in \text{ob}\,\mathcal{P}$, and let $F_Y = [\ , Y] : \mathcal{P} \to$ **p Set**$^{\text{op}}$. If $(\mathcal{T}, \mathcal{P})$ is a Čech extension pair then*

$$[\ , Y] : \mathcal{T} \to \mathbf{p\,Set}^{\text{op}}$$

is a Kan extension of F_Y.

In general this is not the case for non-Čech extension pairs. For example, if $\mathscr{P} = \mathbf{f\,Pol}$ and $Y = S^1$ then $\check{F}_{S^1} = \check{H}^1_f$ which, as is well known, is not a homotopy functor if $\mathscr{T}$ contains the real line.

In this section we shall be concerned with certain representable functors, whose Kan extensions over non-Čech extension pairs are homotopy functors. We begin with a representation theorem.

3.2. Theorem. *Let Y be a polyhedron. Then:*

$$F^K_Y = [\beta -, Y]$$

is a Kan extension of $[-, Y]$ *from* **f Pol** *to* **C Reg T₂**.
(Here, again, β is the Stone–Čech functor.)

Proof. If $Y \in \mathbf{f\,Pol}$ then the theorem follows from 3.1 above and 1.14.

In general; let $Y = \bigcup_\alpha Y_\alpha$ where Y_α ranges over the finite subcomplexes of Y. Since Kan extension commutes with direct limit $F^K_Y = \varinjlim_\alpha [\beta X, Y_\alpha]$. But, for $X \in \mathbf{C\,Reg\,T_2}$, βX is compact so $\varinjlim_\alpha [\beta X, Y_\alpha] = [\beta X, Y]$.

The results in the remainder of this section will depend upon 3.2. Hence from now on we will consider only the case $\mathscr{P} = \mathbf{f\,Pol}$ and use $\check{F}$ to denote the Čech (=Kan) extension of $F : \mathbf{f\,Pol} \to \mathbf{p\,Set}^{\mathrm{op}}$ to $\mathbf{C\,Reg\,T_2}$.

3.2 reduces the study of Čech extensions of homotopy functors to functors of the form $[\beta -, Y]$. These were studied in (2, 3] and we recall the relevant results:

3.3. Theorem [2, 3]. *Let* $\beta^* : [\beta X, Y] \to [X, Y]$ *be the function induced by the inclusion* $X \to \beta X$.

(a) *If X is a finite dimensional normal space and Y has the homotopy type of a CW-Complex of finite type (finite number of cells in each dimension) with* $\pi_1(Y)$ *finite then* β^* *is a bijection.*

(b) *If Y has the homotopy type of a CW-Complex and* ΩY*, the loop space of Y, has the homotopy type of a compact space, then for any X,* β^* *is an injection, and hence* $[\beta -, Y]$ *is a homotopy functor.*

(c) *Suppose Y has the homotopy type of a finite complex with non zero homology. Then* $[\beta -, Y]$ *is not a homotopy functor on the category of metric spaces.*

If $Y = K(G, n)$ is an Eilenberg–MacLane space then $\check{F}_Y = \check{H}^n_f(-; G)$ is the n-th Čech Cohomology functor based on finite covers with coefficient group G. As remarked, $\check{H}^1_f(\ ; Z)$ is not a homotopy functor on any category including the real line. But 3.3a) implies that $\check{H}^n_f(\ ; G)$ is a homotopy functor on **fd Norm** for G finitely generated abelian and $n > 1$. Further 3.3b) implies that $\check{H}^2_f(-; Z)$ is a homotopy functor on all of $\mathbf{C\,Reg\,T_2}$, since $\Omega K(Z, 2)$ has the homotopy type of S^1.

We have in [3] overlooked the following observation.

3.4. Theorem. *$\check{H}^1_f(-; G)$ is a homotopy functor on* $\mathbf{C\,Reg\,T_2}$ *if and only if G is torsion.*

Proof. By [8], $[-, K(G, n)] = \check{H}^n(-; G)$, the n-th Čech cohomology functor based on numerable covers. But, for compact spaces $\check{H}^n(-; G)$ satisfies the universal coefficient theorem [15, p. 336] so:

$$\check{H}^n(\beta X; G) \cong (\check{H}^n(\beta X) \otimes G) \oplus (\mathrm{Tor}(\check{H}^{n+1}(\beta X); G)).$$

In [2] it is shown that $\check{H}^1(\beta X) \cong \check{H}^1(X) \oplus D$ where D is a torsion less divisible group. (The result in [2] is stated only for normal spaces, but the proof clearly goes through for completely regular spaces). Hence, if G is a torsion abelian group (i.e. has no elements of infinite order) then:

$$\check{H}^1(\beta X) \otimes G \cong (\check{H}^1(X) \oplus D) \otimes G \cong H^1(X) \otimes G.$$

But by the argument above $\check{H}^2 \circ \beta = \check{H}^2_f$ is a homotopy functor on $\mathbf{C\,Reg\,T_2}$, and therefore $\check{H}^1_f(-; G)$ is a homotopy functor $\mathbf{C\,Reg\,T_2}$.

On the other hand, it is not hard to show that $\check{H}^1_f(R^1; G) \neq 0$ if G contains an element of infinite order [1].

For $n > 1$, we have no such complete answer as to when $\check{H}^n_f(\ ; G)$ is a homotopy functor on $\mathbf{C\,Reg\,T_2}$. However, we do have the following partial results.

3.5. Theorem. (a) For n odd and G a finitely generated abelian group $\check{H}^n_f(-; G)$ is not a homotopy functor on countable simplicial complexes.

(b) $\check{H}^n_f(-; Q)$ is a homotopy functor on $\mathbf{C\,Reg\,T}$ if and only if n is even.

Proof. (a) Let PS^n be the simplicial path space of S^n[13]. We show that $\check{H}^n_f(PS^n; G) \neq 0$ using a theorem of Weingram [1[illegible]] which says that for G finitely generated abelian and n odd, there is no homotopically non trivial map $\Omega S^n \to K(G, n-1)$ whose image is contained in a compact subset. Since PS^n is contractable this will prove (a). Also, since a well known simple argument shows Weingram's theorem to be true for $G = Q$, we have the "only if" part of (b).

Consider the commutative diagram;

$$\begin{tikzcd}
\Omega S^n \arrow[r] \arrow[d] & K(G, n-1) \arrow[d] \\
PS^n \arrow[r] \arrow[d, "p'"'] & PK(G, n) \arrow[d] \\
S^n \arrow[r, "f"] & K(G, n)
\end{tikzcd}$$

where f corresponds to a non-zero element of the homotopy group $\pi_n(K(G, n))$.

Suppose $\check{H}^n_f(PS^n; G)=[\beta PS^n, K(G,n)]=0$. Then, $\beta(fp')$ is homotopically trivial, so we may lift it to a map a map $\hat{f}: \beta PS^n \to PK(G,n)$. Let $\tilde{f}=\hat{f}|PS^n$. Then:

$$\begin{tikzcd}
\Omega S^n \arrow[r, "\bar{f}"] \arrow[d] & K(G,n-1) \arrow[d] \\
PS^n \arrow[r, "\tilde{f}"] \arrow[d] & PK(G,n) \arrow[d] \\
S^n \arrow[r, "f"] & K(G,n)
\end{tikzcd}$$

commutes with $\bar{f}=\tilde{f}|\Omega S^n$.

From the induced maps of homotopy sequences of fibrations, it is easily seen that $\bar{f}$ is essential but $\bar{f}(\Omega S^n)\subset\hat{f}(\beta\Omega S^n)$ which contradicts Weingram's theorem.

(b) We do not know a direct proof of the "if" part of (b). However, it will follow as a trivial consequence of our study of Čech extensions of K-Theory from finite complexes, which we now begin.

We work inside **C Reg T₂** in order to use 2.12, but in fact, it is a very convenient category, as we will want to form suspensions and cofibration sequences, all of which can be done within this category.

We begin by extending 3.3(b). Rather than letting our notation become too contrary, we state our results in the language of cofunctors. We now give a definition which differs slightly from the usual one.

3.6. Definition. A homotopy cofunctor F: $\mathbf{C\,Reg\,T_2} \to \mathbf{P\,Set}$ is called *half-exact* if for any closed pair $i: A\subset X$,

$$F(A)\xleftarrow{F(i)}F(X)\xleftarrow{(F(j)}F(X\cup_i CA) \text{ is exact,}$$

where $X\cup_i CA$ is the mapping cone of i.

3.7. Theorem. *Let $Y=\bigcup_{i=0}^{\infty} Y_i$ in the weak topology. Suppose $\beta^*:[\beta-, Y_i]\to[-, Y_i]$ is a monomorphism for each i. Then $[\beta-, Y]$ is a half-exact homotopy cofunctor on* $\mathbf{C\,Reg\,T_2}$.

Proof. Firstly, note that we do not claim $\beta^*:[\beta-, Y]\to[-, Y]$ is a monomorphism.

Since the topology on Y is the weak topology, we have $[\beta X, Y]=\varinjlim\,[\beta X, Y_i]$. Hence, there is the monomorphism

$$\varinjlim \beta^*:[\beta-, Y]\to \lim\,[-, Y_i].$$

Thus, as before, since $\varinjlim\,[-, Y_i]$ is a homotopy functor so is $[\beta-, Y]$.

Now consider the following diagram.

$$\begin{tikzcd}
{[\beta A, Y]} \arrow[d, tail] & {[\beta X, Y]} \arrow[l, "\beta(i)^*"'] \arrow[d, tail] & {[\beta(X \cup_i CA), Y]} \arrow[l, "\beta(j)^*"'] \arrow[d, tail] \\
\varprojlim\, [A, Y_i] & \varprojlim\, [X, Y_i] \arrow[l, "i^*"'] & \varprojlim\, [X \cup_i CA, Y_i] \arrow[l, "j^*"']
\end{tikzcd}$$

Since the bottom now is exact and the vertical maps are monomorphisms, we have $\beta(j)^*\beta(i)^* = 0$.

Now let $f: \beta X \to Y$ be such that $f \mid \beta A$ is homotopic to the constant map.

Let $\tilde{g}: \beta X \cup_{\beta(i)} C(\beta A) \to Y$ be the associated extension of f. Since $\beta X \cup_{\beta(i)} C(\beta A)$ is compact we have the following diagram:

$$\begin{tikzcd}
\beta(X \cup_i CA) \arrow[dr] & \\
& \beta X \cup_{\beta(i)} C(\beta A) \\
X \cup_i CA \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Define $g = \tilde{g}\sigma$. One checks $g\beta(j) = f$.

We are particularly interested in the following application.

3.8. Corollary. *Let $G = \bigcup_{i=1}^{\infty} G_i$ where $G_i \subseteq G_{i+1}$ is a sequence of compact groups. Then if $BG = \bigcup_{i=1}^{\infty} BG_i$, the union of the classifying spaces BG_i we have that $[\beta -, BG]$ is a half-exact homotopy cofunctor.*

3.9. The two particular examples we intend to study are $U = \bigcup_{i=1}^{\infty} U(n)$ and $O = \bigcup_{i=1}^{\infty} O(n)$ the unitary and orthogonal groups. We denote by $\check{K}_f$ and $\check{K}O_f$ the right Kan extensions of $\tilde{K}$ and $\tilde{K}O$ from **f Pol** to **C Reg T₂**. Of course, $\tilde{K}$ and $\tilde{K}O_f$ are classified on **f Pol** by BU and BO respectively. Hence, $\check{K}_f$ and $\check{K}O_f$ are half-exact homotopy cofunctors on $\mathbf{C\,Reg\,T_2}$. We now show that some of the formal structure associated with $\tilde{K}$ and $\tilde{K}O$ extend to $\check{K}_f$ and $\check{K}O_f$. We restrict attention to $\check{K}_f$, first observing that $\check{K}_f$ is not representable.

3.10. Theorem. *The functor $\check{K}_f$ is not representable.*

Proof. Unraveling the definition of representability, if $\check{K}_f = [\beta -, BU]$ were representable there would be a space $Y \in \mathbf{C\,Reg\,T_2}$ and a map $u: \beta Y \to BU$ such that the map

$$u_*: [X, Y] \to [\beta X, BU]$$

given by $u_*[f] \to [u\beta(f)]$ is at least a set isomorphism. Letting $\tilde{u} = u \cdot i: Y \xrightarrow{i} \beta Y \xrightarrow{u} BU$ and X be the n-sphere, we would have $\tilde{u}_*: \pi_n(Y) \cong \pi_n(BU)$ for all n.

On the other hand, since βY is compact, we have that the image of Y lies in a compact subspace of BU which by the Whitehead theorem would imply that the homology of BU is finite dimensional.

Now let $\mathrm{ch}: \tilde{K}_f(-)\otimes Q \cong \sum_{n=1}^{\infty} \oplus H_f^{2n}(-; Q)$ be the Chern character [16] restricted to finite complexes. Let $\check{\mathrm{c}}\mathrm{h}$ be the induced natural transformation on the Kan extension.

3.11. Theorem. *On* **C Reg T$_2$**

$$\check{\mathrm{c}}\mathrm{h}: \check{K}_f(-)\otimes Q \cong \sum_{n=1}^{\infty} \oplus \check{H}_f^{2n}(-; Q).$$

Proof. That ch extends to a natural isomorphism is trivial. Unfortunately, the extensions are slightly wrong. In particular we actually have:

$$\check{\mathrm{c}}\mathrm{h}: (\tilde{K}_f(-)\otimes Q)^K \cong \left(\sum_{n=1}^{\infty} \oplus H_f^{2n}(-, Q)\right)^K.$$

To complete the proof one observes that right Kan extensions commute with direct sums and tensor products.

3.12. Proof of (3.5)(b). Since $\check{K}_f$ is a homotopy functor on **C Reg T$_2$**, so is $\check{K}_f(-1)\otimes Q$. Hence, by (3.9) so is $\sum_{n=1}^{\infty} \oplus \check{H}_f^{2n}(-, Q)$ and finally $\check{H}_f^{2n}(-, Q)$.

3.13. Final Remarks. (3.5) (b) is particularly interesting from our point of view since $K(Q, 2n)$ and $K(G, 1)$, G an infinite direct limit of finite group, are not of finite type. Hence, one cannot conclude that the corresponding Kan extensions are homotopy functors on finite dimensional normal spaces by applying (3.3)(a). On the other hand, by (3.3(a) we know $\check{H}_f^{2n+1}(-, Z)$ is a homotopy functor on finite dimensional normal space but not on **C Reg T$_2$**. (In fact, our example is a countable CW Complex).

The general problem on when $\check{H}_f^n(-, G)$, G arbitrary, is a homotopy functor would therefore, seem to require a more definitive technical result than either (3.3) or (3.8).

The functors $\check{K}_f$ and $\check{K}O_f$ also seem to be worth further investigation. If, as usual, one defines $\check{K}_f^{-n}(X) = \check{K}_f(S^n X)$ then by (3.8) and the Puppe sequence one has the usual long exact sequence.

$$\check{K}_f^0(A) \leftarrow \check{K}_f^0(X) \leftarrow \check{K}_f^0(X \cup CA) \leftarrow \check{K}_f^{-1}(A) \leftarrow \cdots \check{K}_f^{-n}(X) \leftarrow \tag{3.14}$$

On the positive side if $A \subseteq X$ are a finite dimensional normal pair, then this sequence coincides with the usual exact sequence of K-theory. On **C Reg T$_2$** this is not the case. To see this, one observes that by the proof of (3.7) one would first of all need $\varinjlim [X, BU(n)] \to [X, BU]$ to be onto for all X in **C Reg T₂**. Letting $X = BU$, we see that this is not the case in general.

We also do not know if $\check{K}_f(X) \cong \check{K}_f(S^2X)$. We suspect that this not the case. The problem seems to be that Kan extension does not commute with suspension. In particular using the methods of [1] one can show that $\check{K}_f^{-1}$ is not even a homotopy functor on finite dimensional normal spaces. Without Bott periodicity, we are unable to extend (3.14) to the left.

One approach to this particular question is to "force" periodicity. We observe that the Bott isomorphism on **f Pol**:

$$\mathrm{bott}: \check{K}_f \cong \check{K}_f \circ S^2$$

passes to a natural transformation

$$\check{\mathrm{bott}}: \check{K}_f \to \check{K}_f \circ S^2.$$

We can define: $\hat{K} = \varinjlim \check{K}_f \circ S^{2n}$.

This gives a full cohomology theory that again agrees with the usual one on **fd Norm** and again is not representable.

Finally, we note on passing that 3.1–3 hold for **Set** valued functors as well. Hence, some of the above is available in slightly more generality then actually stated.

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